Solar Rotational Tomography

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Solar Rotational Tomography

- What is it?
  - Classical least square minimization
  - Total variation minimization
  - Dynamic reconstruction
The white light Solar Corona
The white light Solar Corona

2 components:
- F corona, is dust diffusion & uniform
- K corona, Thomson scattering by the electrons of the plasma
  - Large scale & stable structures: “streamers”
  - Fast events like CMEs

Tomography
- of the K corona
  - Use polarizers (less frequent for LASCO C2)
  - Or do an F-K separation: more frequent but less accurate (systematics)
- Goal: reconstruct stable large scale
- Use solar proper rotation (~ 1 month)

Difference with usual tomography:
- Occultation due to the sun occulter & telescope field
- Systematics due to coronograph
- Missing angles (few images/day for rotation of ~13deg/day)
- Radial decrease
  - Thomson scattering
  - Intrinsic decrease of Ne(r) with r
- Rotation axis not perpendicular
- Differential rotation?
- Time variability
Thomson scattering along the LOS

Thomson scattering
- Decrease with $r$
- Change with diffusion angle

![Graph showing Thomson scattering along the LOS](image)
Rotation axis not perpendicular

=> full 3D reconstruction

=> missing Fourier modes
A Linear Problem

- Thomson scattering equation

\[ y_j = B_\odot \int n_e(l_j)Q[r(l_j), \theta(l_j)] \, dl_j + n_j \]

- A Linear problem

\[ y = Ax + n \]

- Linear Least square solution

\[ \hat{x} = \arg\min_x \| y - Ax \|_2^2 \]

\[ \hat{x} = (A^t A)^{-1} A^t y \]

- Iterative LSQR algorithm
Ill-posed problem

Original

Least square solution without regularization
Tikhonov regularization

**Principle**

\[
\hat{x} = \arg\min_x \left\{ \| y - Ax \|_2^2 + \lambda^2 \| W x \|_2^2 \right\}
\]

\[
\hat{x} = \arg\min_x \left\| \begin{pmatrix} A \\ \lambda W \end{pmatrix} x - \begin{pmatrix} y \\ 0 \end{pmatrix} \right\|_2^2
\]

**Choice of regularization operator**

\[
W = \begin{pmatrix}
\frac{\partial^2}{\partial \theta^2} \\
\frac{\partial^2}{\partial \phi^2}
\end{pmatrix}
\]
Choose the right regularization parameter

- Start with operator norm ratio
  \[ \lambda = \frac{\|A\|}{\|W\|} \]

- L-curve
  [Graph showing L-curve with corner at 11]

- GCV
  \[ \arg\min_{\lambda} \frac{m\| (I - K)y \|_2^2}{(\text{trace}(I - K))^2} \]

  with
  \[ K = A(A^tA + \lambda^2W^tW)^{-1}A^t \]
Tikhonov regularization results
model - 14 images

lambda=1e-5, snr=2

GCV, lambda=8.6e-7, snr=2.6
Tikhonov regularization results
model - 77 images

GCV, $\lambda=8.6e-7$, SNR=3.5
Tikhonov regularization results
lasco images, June 2006

\( \text{lamb}da = 1e^{-5} \)

\( \text{LOG10}(Ne) \ r(8) = 3.0125 \)

\( \text{LOG10}(Ne) \ r(29) = 5.0075 \)

\( \text{gcv, lamb}da = 7e^{-7} \)

\( \text{LOG10}(Ne) \ r(8) = 3.0125 \)

\( \text{LOG10}(Ne) \ r(29) = 5.0075 \)
Total variation minimization

Goal is to preserve high frequencies components

$$\hat{x} = \text{argmin}_x \left\{ \|y - Ax\|_2^2 + \lambda^2 \|\nabla x\|_1 \right\}$$

With gradient only in theta-phi plane: $$\nabla = \begin{pmatrix} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}$$

We will actually directly solve (basis pursuit denoising)

$$\hat{x} = \text{argmin}_{x > 0} \left\{ \|\nabla x\|_1 \right\} \quad \|y - Ax\|_2^2 < \epsilon$$
The TFOCS package

Templates for First-Order Conic Solvers
Stephen Becker, Emmanuel J. Candes and Michael Grant, 2010
http://tfocs.stanford.edu/

1/ determine a conic formulation of the problem

\[
\text{minimize} \quad f(x)
\]
\[
\text{subject to} \quad A(x) + b \in \mathcal{K}.
\]

2/ determine its dual

\[
\text{maximize} \quad g(\lambda)
\]
\[
\text{subject to} \quad \lambda \in \mathcal{K}^*,
\]
\[
g(\lambda) \triangleq \inf_x L(x, \lambda) = \inf_x f(x) - \langle \lambda, A(x) + b \rangle
\]
\[
\mathcal{K}^* = \{ \lambda \in \mathbb{R}^m : \langle \lambda, x \rangle \geq 0 \text{ for all } x \in \mathcal{K} \}
\]

3/ apply smoothing (Nesterov 2005)

\[
\text{minimize} \quad f_\mu(x) \triangleq f(x) + \mu d(x)
\]
\[
\text{subject to} \quad A(x) + b \in \mathcal{K},
\]

4/ solve using an optimal first-order method
A. Auslender and M. Teboulle, 2006
TFOCS, first try: positivity constraint

Positivity constraint on the L2 regularized least square:

\[ \hat{x} = \arg\min_{x>0} \left\{ \|y - Ax\|^2_2 + \lambda^2 \|Wx\|^2_2 \right\} \]

Can be written

\[ \hat{x} = \arg\min_x f(Ax + b) + h(x) \]

where \( f \) is quadratic form (for L2 norm)
where \( h(x) = 0 \) if \( x > 0 \) and \( h(x) = +\infty \) otherwise

\[
A = \begin{pmatrix} A \\ \lambda W \end{pmatrix} \quad b = \begin{pmatrix} -y \\ 0 \end{pmatrix}
\]

\[ x = \text{tfocs}(\text{smooth}_\text{quad}, \{ A, -y; \lambda \text{m} \text{ba}^*W, 0 \}, \text{proj}_\text{Rplus}()) ; \]
TFOCS total variation, BPDN

Solve  \( \hat{x} = \arg\min_{x > 0} \| \nabla x \|_1 \) \\
\( \| y - Ax \|_2^2 < \epsilon \)

Rewrite  \( \hat{x} = \arg\min_{x} t \)
\( \| \nabla x \|_1 < t \)
\( \| y - Ax \|_2^2 < \epsilon \)
\( \| x \| > 0 \)

Then write the dual.

BPDNW solver:

\( x = \text{tfocs\_SCD}( \text{proj\_Rplus}(), \{ A, -y; W, 0 \}, \{ \text{prox\_l2( epsilon )}, \text{proj\_linf(proxScale) } \}, \mu, \text{vec}(x0), z0 \); \)
total variation results
model - 14 images

TV, snr=2.1

Thikonov L2 norm, snr=2.6
total variation results
lasco – june 2006 - 77 images

TV

Thikonov, lambda=7e-7
Dynamic SRT

Concept: add the time dimension to the object to reconstruct

History: One attempt with Kalman filtering (Buttala et al.), with Identity as the linear state-transition function

But difference with usual dynamic tomography:
  Use the same set of data than static reconstruction.
  => very undetermined system, needs stronger regularization

\[ A_t = \begin{pmatrix}
  A(\theta = 1) & 0 & 0 & \cdots & 0 \\
  0 & A(\theta = 2) & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  \vdots & \vdots & \ddots & A(\theta = i) & \vdots \\
  \vdots & \vdots & \vdots & \vdots & A(\theta = 20)
\end{pmatrix} \]
Dynamic SRT: spatio-temporal constraint

We choose a formally simpler approach than Kalman: multiple constraints:

- **L2 norm:**

  \[
  \hat{x} = \arg\min_x \left\{ \| y - Ax \|_2^2 + \lambda^2 \| W x \|_2^2 + \mu^2 \| \nabla_t x \|_2^2 \right\} \\
  W = \begin{pmatrix}
  \frac{\partial^2}{\partial \theta^2} \\
  \frac{\partial^2}{\partial \phi^2}
\end{pmatrix}
\]

- **L1 norm**

  \[
  \hat{x} = \arg\min_{x>0} \left\{ \alpha \| \nabla_{\theta,\phi} x \|_1 + \beta \| \nabla_t x \|_1 \right\}
  \| y - Ax \|_2^2 < \epsilon
\]
How to balance the spatial and temporal regularization parameters?

- $\lambda = 1 \times 10^{-5}$
- $\mu = 1 \times 10^{-4}$

- $\lambda = 6 \times 10^{-7}$
- $\mu = 1.2 \times 10^{-7}$
Spatio-temporal total variation regularization
Conclusion

- L1 norm minimization IS tractable for large scale 4D (3D + time) tomography
- Dynamic SRT needs more investigation with 4D models

Perspective?
  - Change grid sampling, equal pixel area on the sphere
  - Sparse decomposition on a curvelet basis (BPDN analysis with TFOCS)
  - Process 15 years of LASCO images to build reference Ne measures