Back to normal

Box-Cox transformations in cosmology

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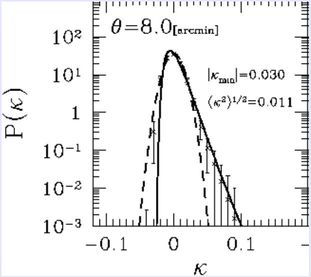
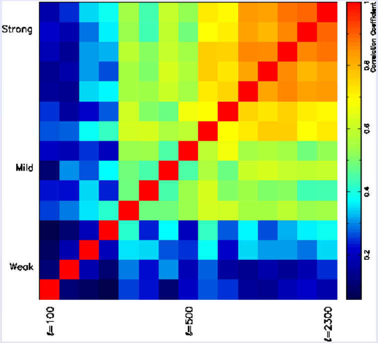
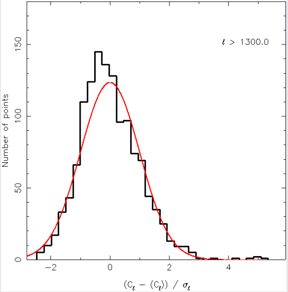
Non-Gaussianity abounds in weak lensing data



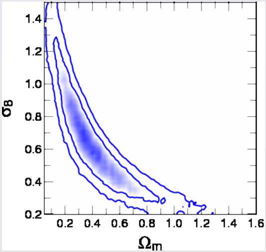
Taruya et al. (2002)



Kiessling et al. (2011)



Kiessling et al. (2011)



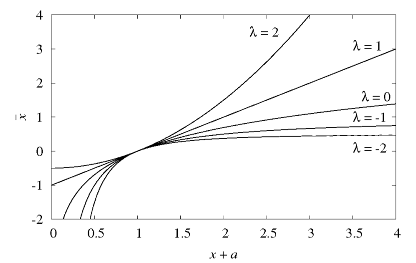
Schrabback et al. (2010)

Contents

Together with: Andy Taylor (IfA), Alina Kiessling (Caltech)

The Box-Cox transformation

- proposed by *Box & Cox (1964)*
- in astronomy: only applied in CMB analysis by *Dineen & Coles (2005)*



Goal:

- transform data set $\boldsymbol{x} \rightarrow \bar{\boldsymbol{x}}$ such that $\bar{\boldsymbol{x}}$ is Gaussian distributed

For every element x_i in the data set

$$\bar{x}_i(\lambda, a) = \begin{cases} \left[(x_i + a)^\lambda - 1\right]/\lambda & \lambda \neq 0 \\ \ln(x_i + a) & \lambda = 0 \end{cases}$$

Box-Cox parameters: λ, a

Determining the Box-Cox parameters

Maximum likelihood (ML) approach:

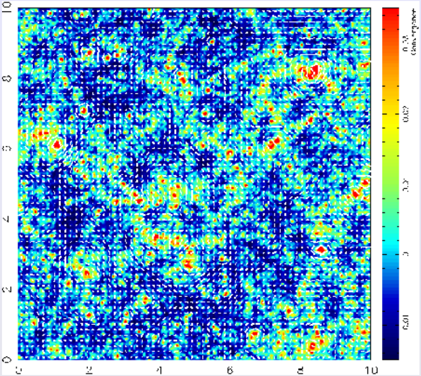
$$L(\boldsymbol{x}) = \bar{L}(\bar{\boldsymbol{x}}) \, \frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}} = \bar{L}(\bar{\boldsymbol{x}}) \, \prod_{i=1}^{n} [x_i + a]^{\lambda - 1}$$

- transformation should ensure that $\bar{L}(\bar{\boldsymbol{x}})$ is a Gaussian
- Free parameters in $L(\boldsymbol{x})$: λ, a, mean and covariance of Gaussian $\bar{L}(\bar{\boldsymbol{x}})$
- compare this model for $L(\boldsymbol{x})$ to the data:
  insert ML estimates of mean and covariance into expression for $L(\boldsymbol{x})$
  $\rightarrow$ Marginal log-likelihood

$$\mathcal{L}(\lambda, a) = -\frac{n}{2} \, \ln \det \mathrm{Cov} \left[\bar{\hat{x}}(\lambda, a) \right]_{\mathrm{ML}} + (\lambda - 1) \sum_{i=1}^{n} \ln(\hat{x}_i + a)$$

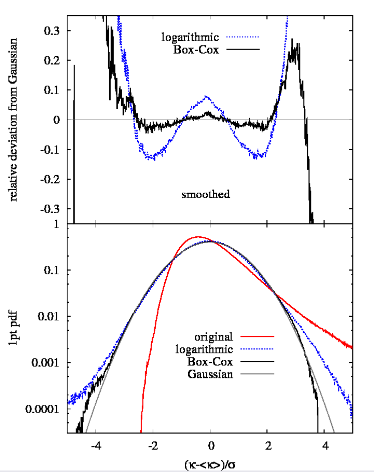
- maximise with respect to $\lambda, a \rightarrow$ optimal Box-Cox parameters
- straightforward generalisation to higher-dimensional data
  (pair of λ, a for each dimension)
- Bayesian approach yields very similar result *(Box & Cox 1964)*

The convergence distribution



shear/convergence field

Kiessling et al. (2011)



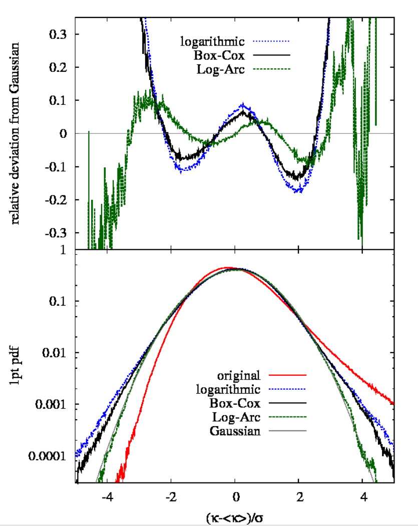
1point probability distribution for κ

Joachimi et al. (2011)

$\rightarrow$ study only the 1point distribution – simplification!

$\rightarrow$ log-transform performs fairly well, but not optimal ($\lambda = -2.2$; $a = 0.08$)

Gaussianisation in presence of shape noise



1point probability distribution for κ now with realistic level of Gaussian shape noise added

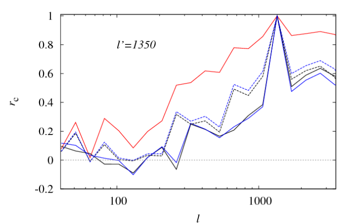
Log-Arc transformation:

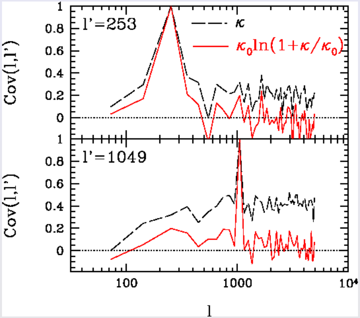
$$\bar{x}(s,a) = \begin{cases} \arctan\left[s \ln(x+a)\right]/s & s \neq 0 \\ \ln(x+a) & s = 0 \end{cases}$$

$\rightarrow$ Box-Cox formalism is versatile!

type	skew.	kurt.	$D_{KL} * 100$
original	0.83	3.11	3.8
log	-0.05	0.73	0.7
Box-Cox	-0.02	0.43	0.4
Log-Arc	-0.05	-0.02	0.2

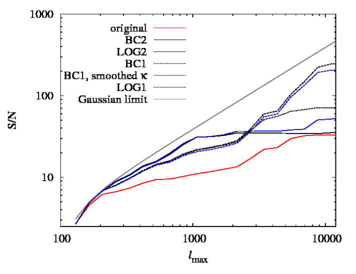
Covariances & signal-to-noise

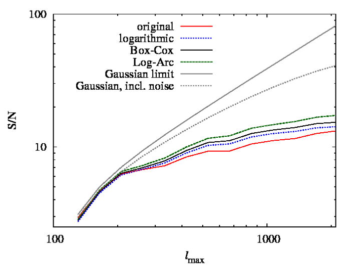




Top: $\sigma_8 = 0.81$; *Joachimi et al. (2011)*

Bottom: $\sigma_8 = 0.76$; *Seo et al. (2011a)*





Top: S/N without shape noise

Bottom: S/N with shape noise

Transformed power spectrum

Taylor-expand:

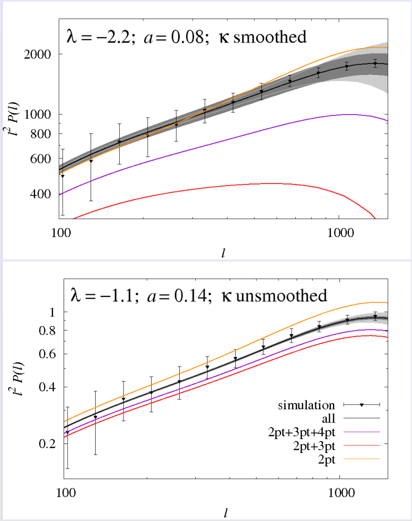
$$(\kappa + a)^\lambda = a^\lambda + \lambda a^{\lambda-1}\kappa + \frac{\lambda(\lambda-1)}{2} a^{\lambda-2}\kappa^2 + \dots$$

Express transformed power spectrum $P_{\bar\kappa}$ in terms of κ statistics:

$$\begin{aligned} P_{\bar\kappa}(\ell) &= a^{2\lambda-2}\Big\{ P_\kappa(\ell) + (\lambda-1)\, a^{-1} \int \frac{\mathrm{d}^2\ell_1}{(2\pi)^2}\, B_\kappa(\ell, \ell_1, |\boldsymbol{\ell} - \boldsymbol{\ell}_1|) \\ &\quad + (\lambda-1)(\lambda-2)\, a^{-2} P_\kappa(\ell) \int \frac{\mathrm{d}^2\ell_1}{(2\pi)^2}\, P_\kappa(\ell_1) + \dots \end{aligned}$$

- $\ell \to \infty$ in integrals $\to$ need to smooth κ map (use Gaussian kernel)
- simulation discreteness noise and shape noise contribute also to low ℓ in transformed power spectrum $\to$ noise needs to be modelled

Comparison to simulations



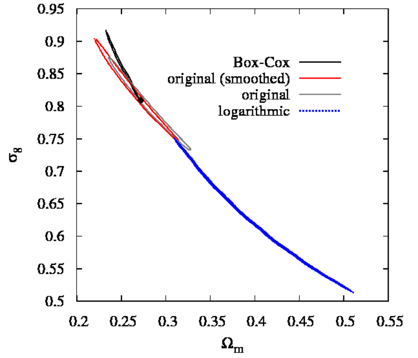
Transformed κ power spectra
light grey: bispectrum uncertainty
dark grey: higher-order uncertainty

- power spectra with non-linear corrections *(Smith et al. 2003)* – accurate to $\ell \sim 1500$
- bispectra from perturbation theory with non-linear corr. *(Scoccimarro & Couchman 2001)*
- (initially) neglect trispectrum contributions to 4th-order term
- smoothing with kernel width 1.5 arcmin
- analytic computation of simulation and shape noise

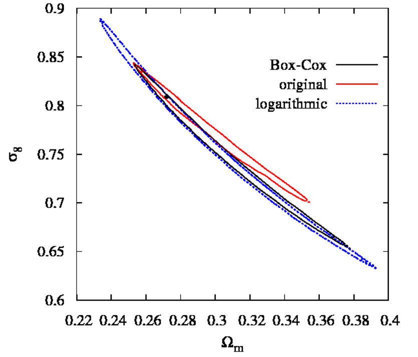
That's not enough!

$\rightarrow$ include up to 6th order in κ, use lognormal model and rescale each κ order with simulation-based moment (factors of a few larger)

Cosmological parameter constraints



noise-free



with shape noise

Disappointing performance because

- only integral over certain triangle, quadrangle, etc. configurations contribute to higher-order statistics
- only sum of different orders enters data vector
- shape noise makes distribution more Gaussian $\rightarrow$ less to gain

Confirmed by recent work using numerical simulations *(Seo et al. 2011b)*

Box-Cox-Fisher formalism

Fisher information matrix

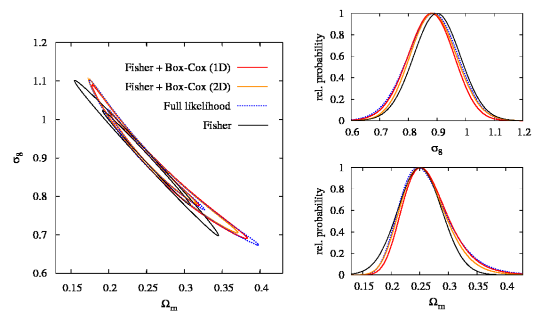
$$F_{\mu\nu} = -\left\langle \frac{\partial^2 \ln L}{\partial p_\mu \partial p_\nu} \right\rangle = \sum_\ell \frac{\partial P_\kappa(\ell)}{\partial p_\mu} \, \mathrm{Cov}^{-1} \left\{ P_\kappa(\ell) \right\} \, \frac{\partial P_\kappa(\ell)}{\partial p_\nu}$$

$\rightarrow$ can only fully represent Gaussian posteriors $\leftrightarrow$ elliptical contours

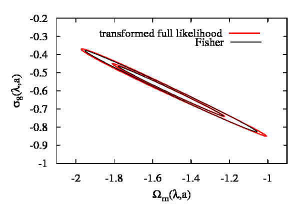
How to get more accurate posterior forecasts:

1. determine Box-Cox parameters for every posterior dimension e.g. via small MCMC sample
2. compute Fisher matrix $\bar{F}$ and posterior maximum $\bar{\boldsymbol{p}}_{\rm max}$ for transformed cosmological parameters (approximation in terms of original F)
3. transformed Gaussian posterior given by $\bar{\mathcal{P}} \propto \exp\left\{ -\frac{1}{2} \left(\bar{\boldsymbol{p}} - \bar{\boldsymbol{p}}_{\rm max} \right)^\tau \, \bar{F} \, \left(\bar{\boldsymbol{p}} - \bar{\boldsymbol{p}}_{\rm max} \right) \right\}$
4. inverse Box-Cox transformation yields original posterior $\mathcal{P}$

Performance



2D and 1D posteriors for $\Omega_{\rm m}$ and σ_8

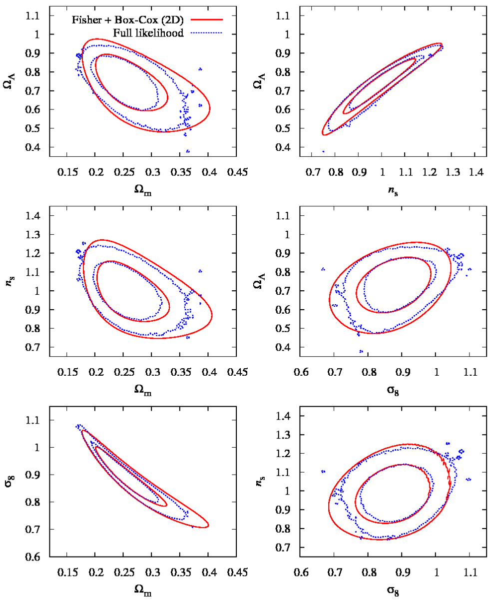


Transformed posterior and transformed Fisher matrix

Joachimi & Taylor (2011)

parameters changed	$D_{\rm KL}(\Omega_{\rm m})$	$D_{\rm KL}(\sigma_8)$	$D_{\rm KL}(\Omega_{\rm m}, \sigma_8)$
Fisher matrix	0.143	0.029	3.482
Box-Cox, fid. parameters	0.013	0.013	0.017
$\Omega_{\rm m}$: 0.25 $\rightarrow$ 0.225	0.008	0.008	0.025
$z_{\rm med}$: 0.9 $\rightarrow$ 1.0; $n_{\rm g}$: 20 $\rightarrow$ 37.4 $'^{-2}$	0.005	0.005	0.019
$\ell_{\rm max}$: 10000 $\rightarrow$ 8700	0.015	0.015	0.019
$A_{\rm s}$: 100 $\rightarrow$ 110 deg^2	0.013	0.012	0.016

A higher-dimensional example

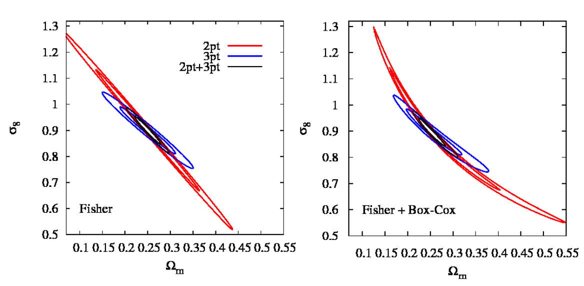


4D cosmic shear posterior $\{\Omega_{\rm m}, \sigma_8, n_{\rm s}, \Omega_\Lambda\}$

Which set of contours is more trustworthy?

Application: Bispectrum constraints

How much do 3pt statistics add to parameter constraints?



forecast of 1σ and 2σ constraints from cosmic shear power spectrum and bispectrum

Joachimi & Taylor (2011)

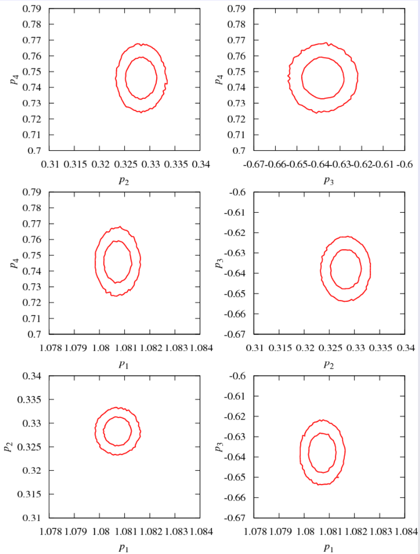
statistics	Fisher	B-C-F
2pt	$0.90^{+0.23}_{-0.23}$	$0.85^{+0.31}_{-0.27}$
3pt	$0.90^{+0.09}_{-0.09}$	$0.89^{+0.11}_{-0.12}$
2pt + 3pt	$0.90^{+0.05}_{-0.05}$	$0.90^{+0.06}_{-0.08}$

Marginal 2σ constraints on σ_8

$\rightarrow$ the stronger the constraints the better is the standard Fisher matrix

But: larger survey $\rightarrow$ more parameters with new degeneracies

Independent parameters for weak lensing



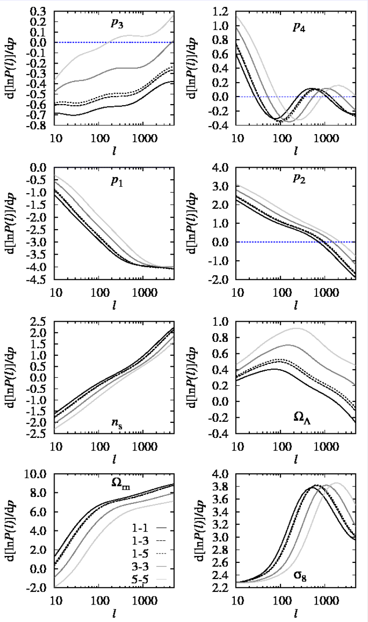
Joachimi & Taylor, in prep.

1. Box-Cox transformation of posterior $\rightarrow$ multivariate Gaussian posterior
2. decorrelation via principal component analysis $\rightarrow$ statistically independent Gaussian parameters

$\Rightarrow$

- speeds up MCMC sampling
- simplifies marginalisation
- highlights where parameter constraints come from

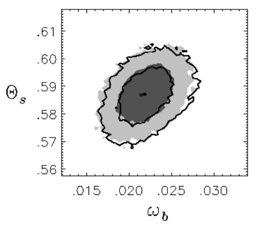
Interpretation of new parameter set



Relative power spectrum derivatives for parameter set $\{\Omega_m, \sigma_8, n_s, \Omega_\Lambda\}$ and corresponding normal parameters $\{p_1, p_2, p_3, p_4\}$

Joachimi & Taylor, in prep.

Can one define physically motivated weak lensing normal parameters in analogy to CMB normal parameters *(Kosowsky et al. 2002, Sandvik et al. 2004)*?



Contours: MCMC
Shaded area: Gaussian approx.
Chu et al. (2003)

Work in progress ...

Conclusions

- Gaussianisation of the 1pt weak lensing convergence does NOT improve constraints on cosmology in cases with realistic noise
- Box-Cox transformations can be combined with Fisher matrices to produce accurate forecasts of posterior distributions
- Gaussianisation of posteriors may yield a recipe to define normal (statistically independent) parameters for arbitrary inference problems

$\rightarrow$ Box-Cox transformations are a simple, effective, and versatile tool to Gaussianise uni- and multivariate distributions

To-do list:

- Go beyond 1pt convergence Gaussianisation;
  treat convergence and shape noise distributions separately
- Investigate alternative ways to efficiently exploit cosmological information in higher-order statistics of large-scale structure
- Find physical weak lensing normal parameters