A Novel Convex Optimization Approach to Optical Interferometric Imaging

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Optical Interferometry

Keck optical interferometer

- Interferometers provide sparse measurements of the Fourier transform of the observed object (complex visibilities)
At Optical wavelengths the complex visibilities cannot be directly measured due to atmospheric turbulences.
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- Optical Interferometric imaging can be seen as a **Partial Phase Retrieval Problem**
Classical Phase Retrieval Approaches

- Iterative algorithm (Fienup 1978):
  - full power spectrum
  - based on POCS
  - convergence for special cases
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  - nonlinear-nonconvex problem
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**Problem**

All algorithms are sensitive to initial solution and don’t converge in general.
The Phase Lift Approach

Motivation
Reformulate a nonlinear model into a linear model to have a convex problem.
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- Nonlinear measurement model

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- Nonlinear measurement model
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- Define the rank one matrix \( X = x x^T \)

- Linear model on \( X \)
  \[ y = \text{diag}(AXA^T) \in \mathbb{C}^M = \mathcal{A}(X) \]
The phase retrieval problem can be reformulated as the following optimization problem:

$$\min_{X \in \mathbb{C}^{N \times N}} \|X\|_* \quad \text{s.t.} \quad \|y - \mathcal{A}(X)\| \leq \epsilon \text{ and } X \succeq 0.$$
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**Theorem (Candes et al. 2011)**

*Consider a complex signal $x \in \mathbb{C}^N$. Suppose that $a_i$ are unitary vectors and uniformly distributed on the unit sphere and $M \approx N \log N$. Then the solution to the trace-minimization program is exact with high probability in the sense that it has a unique solution obeying $X = xx^T$.***
We use phase lift ideas to recover the signal from optical interferometric measurements.
Formulating a linear inverse problem: partial PR problem based on tensor recovery

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- In addition to power spectrum measurements we have partial phase information and sparsity and positivity of the signal as prior information.
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Define the 3D tensor $\mathcal{X} = x \otimes x \otimes x$

The following linear problem can be formulated:

$$y = A(\mathcal{X}) + n$$

with $A(\mathcal{X}) = \mathcal{M}\mathcal{F}(\mathcal{X}) \in \mathbb{C}^M$. 

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Convex Formulation

We recover $x$ by solving

$$\min_{\mathcal{X} \in \mathbb{R}^{N \times N \times N}} \sum_{i=1}^{3} \|C_i(\mathcal{X})\|_* + \lambda \|\mathcal{X}\|_1$$

s.t. $\|y - A(\mathcal{X})\|_2 \leq \epsilon$

$C_i(\mathcal{X}) \succeq 0, \ i = 1, 2, 3$

$\mathcal{X} \geq 0,$

where $C_i(\mathcal{X}) \in \mathbb{C}^{N \times N}$ and

$$[C_1(\mathcal{X})]_{i,j} = \sum_k x_{i,j,k}, \ [C_2(\mathcal{X})]_{i,j} = \sum_k x_{i,k,j}, \ [C_3(\mathcal{X})]_{i,j} = \sum_k x_{k,i,j}. $$
We recover $x$ by solving

$$\min_{X \in \mathbb{R}^{N \times N \times N}} \sum_{i=1}^{3} \| C_i(X) \|_* + \lambda \| X \|_1$$

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- We use the parallel proximal algorithm to solve the problem
To improve the quality of the reconstruction, we promote structured sparsity and low rank of the solution by using a reweighted approach.
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We solve a sequence of weighted problems of the form:

$$\min_{X \in \mathbb{R}^{N \times N \times N}} \sum_{i=1}^{3} \|C_i(X)\|_{w^*} + \lambda \|X\|_{w^1}$$

s.t.  
\[ \|y - A(X)\|_2 \leq \epsilon \]
\[ C_i(X) \succeq 0, i = 1, 2, 3 \]
\[ X \geq 0. \]
$w_{i,j,k}^{(t)} = \frac{\delta(t)}{\delta(t) + (|x_{i,i,i}^{(t-1)}| |x_{j,j,j}^{(t-1)}| |x_{k,k,k}^{(t-1)}|)^{1/3}}$
\( w^{(t)}_{i,j,k} = \frac{\delta^{(t)}}{\delta^{(t)} + (|\mathcal{X}^{(t-1)}_{i,i,i}||\mathcal{X}^{(t-1)}_{j,j,j}||\mathcal{X}^{(t-1)}_{k,k,k}|)^{1/3}} \)

2D illustration
$w_{i,j,k}^{(t)} = \frac{\delta(t)}{\delta(t) + (|\mathcal{X}_{i,i,i}^{(t-1)}| |\mathcal{X}_{j,j,j}^{(t-1)}| |\mathcal{X}_{k,k,k}^{(t-1)}|)^{1/3}}$

2D illustration
Reweighted Example (Noise level=30 dB)

Original

![Original Image]
Reweighted Example (Noise level = 30 dB)

Original

First iteration
Reweighted Example (Noise level=30 dB)

Original

Second iteration
Reweighted Example (Noise level=30 dB)

Original

Third iteration
Reweighted Example (Noise level=30 dB)

Original

Fourth iteration
Reweighted Example (Noise level=30 dB)

Original

Final solution (53 dB)
Noiseless Setting

Phase transition diagram, $4 \times 4$ images
Noisy Setting

Reconstruction quality, $4 \times 4$ images, $K=2$, noise level=30 dB
Noisy Setting

Reconstruction quality, $4 \times 4$ images, $K=3$, noise level=$30$ dB
We have shown that the optical interferometric imaging problem can be formulated as a linear inverse problem and solved using convex optimization tools.

The reweighting process is essential to promote structured sparsity in the solution.

New ways to improve the computational efficiency of the algorithm have to be explored.

Future work should concentrate on comparing the proposed approach to state of the art.
Thank You!